VARIATIONAL PRINCIPLE FOR THREE-DIMENSIONAL STEADY-STATE FLOWS OF AN IDEAL FLUID

(VARIATSIONNYI PRINTSIP DLIA TREKHMERNYKH STATSIONARNYKH TECHENII IDEAL'NOI ZHIDKOSTI)

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It is proved that a steady-state flow has an extremal kinetic energy in comparison with "equivorticity" flows. This result is applied to investigate the stability of steady-state flows: if the extremum is a minimum or a maximum, then the flow is stable, i.e. a small change in the initial velocity field causes only a small change in the velocity field for all time. To determine the nature of the extremum (maximum, minimum, etc.) a second variation is explicitly calculated. For the case of plane flows, sufficient conditions of the stability with respect to small finite perturbations are found. These conditions are close to the necessary ones.

1. Finite-dimensional model. We shall show that the equations of the three-dimensional hydrodynamics of an ideal fluid are infinite-dimensional analog to the following finite-dimensional situation. In the space $\mathbf{x} = x_1, \ldots, x_n$ let there be given a system of ordinary differential equations

$$\mathbf{x}^{\cdot} = f(\mathbf{x}) \qquad (1.1)$$

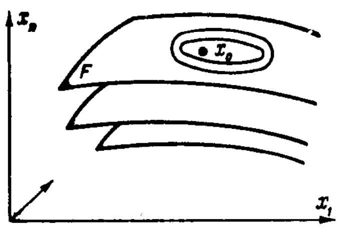

Fig. 1

In addition, we shall assume that a "k-dimensional structure" is given in the space $\mathbf{x}$ (Fig.1), i.e. that the space is decomposed into k-dimensional "sheets" (in the figure: $n = 3$, $k = 2$). We shall assume that the structure is invariant with respect to the system (1.1), i.e. that a trajectory $\mathbf{x}(t)$ which begins on the sheet F still remains on it. We shall call a point $\mathbf{x}$ of the sheet F regular, if in the neighborhood of this point there exists a system of coordinates $y_1, \ldots, y_n$ in which the sheets are given by the equations $y_{k+1} = c_{k+1}, \ldots, y_n = c_n$. In the whole, however, the sheets need not be given by equations (for example, a sheet may be everywhere dense).

We shall assume, finally, that the system (1.1) has a first integral $E(\mathbf{x})$.

We shall consider a local conditional extremum of the function E on the sheet F . We shall assume that it occurs at a regular point $\mathbf{x}_0$ and that the quadratic form d^2E is nonsingular on the sheet F . The following three theorems are easily proved (cf.[1]).

T h e o r e m 1.1. The point $\mathbf{x}_0$ is the equilibrium position of the system (1.1)

$$f(\mathbf{x}_0) = 0$$

T h e o r e m 1.2. If the extremum is a maximum or a minimum, then the equilibrium position $\mathbf{x}_0$ is stable with respect to small finite perturbations.

T h e o r e m 1.3. The spectrum of the problem of small oscillations corresponding to (1.1) $A\xi = \lambda\xi$ $(A = \partial\mathbf{f} / \partial\mathbf{x}$ in $\mathbf{x}_0)$ is symmetric with respect to the real and imaginary axes of λ .

The hydrodynamic analog of these theorems will be formulated below. They are, in fact, corollaries of the general theorems on Li geodesic groups provided partially with an invariant metric (cf.[2]). However, an independent proof is given here which makes use neither of Li groups, nor even of the existence and uniqueness theorems which correspond to the partial differential equations. From a mathematical point of view, these theorems will be "a priori" equalities and estimates.

2. Notation. Let D be a domain, bounded by the fixed surface Γ , in a three-dimensional Euclidean space; let $\mathbf{v}$ be the velocity field of an ideal fluid (incompressible, with density equal to 1 , inviscid, exterior to a nonpotential mass force field) which fills the volume D ; and let p be the pressure.

The Euler equation

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\operatorname{grad} p, \qquad \operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma \qquad (2.1)$$

has as a corollary the Bernoulli equation

$$\frac{\partial\mathbf{v}}{\partial t} = \mathbf{v}\times\mathbf{r} - \operatorname{grad}\lambda, \qquad \mathbf{r} = \operatorname{rot}\mathbf{v}, \qquad \lambda = p + {}^1/_2\mathbf{v}^2 \qquad (2.2)$$

Hence, in view of the identity

$$\operatorname{rot}(\mathbf{A}\times\mathbf{B}) = \{\mathbf{AB}\} + \mathbf{A}\operatorname{div}\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{A} \qquad (2.3)$$

there follows

$$\partial\mathbf{r} / \partial t = \{\mathbf{vr}\} \qquad (2.4)$$

Here $\{\mathbf{AB}\}$ is the Poisson bracket of the vector fields

$$\{\mathbf{AB}\}_i = \sum (\partial A_i / \partial x_j)\, B_j - (\partial B_i / \partial x_j)\, A_j$$

If g is a smooth mapping of $x \to g(x)$, then we shall denote by g^* the corresponding mapping of the vectors

$$(g^*\xi)_i = \sum (\partial g_i / \partial x_j)\, \xi_j$$

3. Equivorticity fields. In order to formulate the law of coservation of vorticity in a form suitable for later use, we shall consider two vector fields $\mathbf{v}$ and $\mathbf{v}'$ in D

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{div} \mathbf{v}' = 0, \quad (\mathbf{v}\cdot\mathbf{n}) = 0, \quad \mathbf{v}'\cdot\mathbf{n} = 0 \quad \text{on } \Gamma$$

D e f i n i t i o n 3.1. The fields $\mathbf{v}$ and $\mathbf{v}'$ are equivorticity fields if there exists a smooth, volume preserving, mapping g of the domain D into itself such that (*)

$$\oint_{\gamma} \mathbf{v}\, dx = \oint_{g\gamma} \mathbf{v}'\, dx \tag{3.1}$$

for every closed contour γ in the domain D.

The law of conservation of vorticity now takes the following form. Let $\mathbf{v}(\mathbf{x},t)$ be the velocity field of the ideal fluid of (2.1).

T h e o r e m 3.1. The fields $\mathbf{v}(\mathbf{x},0)$ and $\mathbf{v}(\mathbf{x},t)$ are fields of equivorticity. In fact, let $\mathbf{x}(t)$ be the trajectory of a fluid particle. The mapping g is then that which transforms $\mathbf{x}(0)$ into $\mathbf{x}(t)$.

We shall now consider the Euler equation (2.1) as the system (1.1) in an infinite-dimensional space of the vector fields $\mathbf{v}(\mathbf{x})$ (where $\operatorname{div} \mathbf{v} = 0$ and $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ). We shall show that this system has the characteristics of the system (1.1). In the space of the fields $\mathbf{v}(\mathbf{x})$ the following structure is specified: two fields belong to the same sheet if they are equivorticity fields. According to Theorem 3.1, this structure is invariant. Steady-state flows are "equilibrium position" of the system. Finally, the Euler equation (2.1) has a first integral of the energy

$$2E = \iiint \mathbf{v}^2\, dx$$

In order to transfer the results of Section 1 to the hydrodynamic equations (2.1) it is necessary to calculate the first and second variations of the function E on the sheet F.

4. Variational principle (**). The following fundamental Theorem is the analog of Theorem 1.1.

T h e o r e m 4.1. The steady-state flow $\mathbf{v}(\mathbf{x})$ has an extremal energy in comparison with all close equivorticity flows $\mathbf{v}'(\mathbf{x})$.

By closeness here is meant closeness "with respect to the sheet", i.e.

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*) The mapping g transforms the vorticity of $\mathbf{v}$ into vorticity of $\mathbf{v}'$

$$g^*\operatorname{rot} \mathbf{v} = \operatorname{rot} \mathbf{v}' \tag{3.2}$$

Actually, if ξ, η is an infinitesimal parallelogram, then, since $\det g^* = 1$,

$$\xi\times\eta\cdot\operatorname{rot} \mathbf{v} = (g^*\xi)\times(g^*\eta)\cdot(g^*\operatorname{rot} \mathbf{v})$$

and corresponding to (3.1)

$$(\xi\times\eta\cdot\operatorname{rot} \mathbf{v}) = (g^*\xi)\times(g^*\eta)\cdot\operatorname{rot} \mathbf{v}'$$

If the domain D is not simply connected, then condition (3.1) is stronger than (3.2).

**) Another variational principle relating to unsteady flows has been determined and applied in the investigation of stability y Fjørtoft [3].

$\mathbf{v}'(\mathbf{x})$ is considered to be close to $\mathbf{v}(\mathbf{x})$ if the corresponding mapping g in (3.1) is close to the identity mapping. To determine the closeness of g to the identity mapping we shall introduce the "coordinates" f into the space g in the following way.

Let $\mathbf{f}(\mathbf{x})$ be a vector field in D such that

$$\operatorname{div}\mathbf{f} = 0, \qquad \mathbf{f}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma$$

Definition 4.1. Lt $g_t = \exp ft$ be a mapping of D into itself, determined by the solutions $\mathbf{x}(t)$ of the ordinary differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ according to Formula $g(\mathbf{x}(0)) = \mathbf{x}(t)$.

The field $\mathbf{v}'$ will be considered to be close to $\mathbf{v}$ if the "coordinates" f of the transformation g in (3.1) are small. In this case, the velocity perturbation $\delta\mathbf{v} = \mathbf{v}' - \mathbf{v}$. is also small. The relation between $\delta\mathbf{v}$ and $\mathbf{f}$ is given by the following Formula (4.2).

Lemma 4.1. If for every closed contour γ in D

$$\oint_{g_{-t}\gamma} \mathbf{v}\,d\mathbf{x} = \oint_{\gamma} \mathbf{v}'\,d\mathbf{x}, \qquad \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{v}' = 0 \tag{4.1}$$

then

$$\mathbf{v}' - \mathbf{v} = t(\mathbf{f}\times\mathbf{r}) + {}^1/_2 t^2\,\mathbf{f}\times\{\mathbf{fr}\} + O(t^3) + \operatorname{grad}\alpha \tag{4.2}$$

where α is a single-valued function and $\mathbf{r} = \operatorname{rot}\mathbf{v}$.

Proof of the Lemma. According to the Stokes formula

$$\frac{d}{dt}\oint_{g_{-t}\gamma} \mathbf{v}\,d\mathbf{x} = -\frac{1}{dt}\iint \mathbf{r}\,ds = \oint_{g_{-t}\gamma} \mathbf{f}\times\mathbf{r}\,d\mathbf{x} \tag{4.3}$$

Since the Jacobian of g_{-t}^* is equal to unity, we then have

$$\oint_{g_{-t}\gamma} \mathbf{f}\times\mathbf{r}\,d\mathbf{x} = \int_{\gamma} g_t^*\mathbf{f}(g_{-t}\mathbf{x})\times g_t^*\mathbf{r}(g_{-t}\mathbf{x})\,d\mathbf{x} \tag{4.4}$$

But, according to the definition of g_t we have $g_t^*\mathbf{f}(g_{-t}\mathbf{x}) = \mathbf{f}(\mathbf{x})$. Therefore, (4.3) and (4.4) give

$$\frac{d}{dt}\oint_{g_{-t}\gamma} \mathbf{v}\,d\mathbf{x} = \oint_{\gamma} \mathbf{f}\times\mathbf{r}(t)\,d\mathbf{x} \tag{4.5}$$

The field $\mathbf{r}(t)$ here is defined by Formula

$$\mathbf{r}(\mathbf{x}, t) = g_t^*\mathbf{r}(g_{-t}\mathbf{x}) \tag{4.6}$$

Differentiating (4.6), we find

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \{\mathbf{fr}\}, \qquad \mathbf{r}(t) = \mathbf{r} + \{\mathbf{fr}\}\,t + O(t^2) \tag{4.7}$$

But, according to condition (4.1),

$$\frac{d}{dt}\oint_{g_{-t}\gamma} \mathbf{v}\,d\mathbf{x} = \oint_{\gamma} \frac{\partial\mathbf{v}'}{\partial t}\,d\mathbf{x}, \qquad \mathbf{v}'|_{t=0} = \mathbf{v} \tag{4.8}$$

Integrating (4.5) and (4.6) with respect to t, we find from (4.6)

$$\oint_{\gamma} (\mathbf{v}' - \mathbf{v})\,d\mathbf{x} = \oint_{\gamma}\int_0^t \mathbf{f}\times[\mathbf{r} + \{\mathbf{f}, \mathbf{r}\}\,t + O(t^2)]\,dt\,d\mathbf{x}$$

which is equivalent to (4.2).

Proof of the basic Theorem. If $\mathbf{v}'$ is a equivorticity flow close to the steady-state flow $\mathbf{v}$, then according to (4.2) the first variation is

$$\delta\mathbf{v} = \mathbf{f}\times\mathbf{r} + \operatorname{grad}\alpha$$

Therefore,

$$\delta E = \iiint \mathbf{v}\cdot\delta\mathbf{v}\,d\mathbf{x} = \iiint \mathbf{v}\cdot(f\times r + \operatorname{grad}\alpha)\,d\mathbf{x} = \iiint [\mathbf{f}\cdot(\mathbf{r}\times\mathbf{v}) + \mathbf{v}\cdot\operatorname{grad}\alpha]\,d\mathbf{x}$$

For a steady-state flow, according to (2.2),

$$\mathbf{r}\times\mathbf{v} = -\operatorname{grad}\lambda$$

Therefore,

$$\delta E = \iiint (\mathbf{v}\cdot\operatorname{grad}\alpha - \mathbf{f}\cdot\operatorname{grad}\lambda)\,d\mathbf{x} = 0$$

This result is obtained by integrating by parts, taking into account Equalities $\operatorname{div}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{f} = 0; \qquad (\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0, \quad (f, n)|_\Gamma = 0$

The Theorem is proved.

5. Formula for the second variation. According to Lemma 4.1

$$\mathbf{v}' = \mathbf{v} + \delta\mathbf{v} + \delta^2\mathbf{v} + O(f^3)$$

where

$$\delta\mathbf{v} = \mathbf{f}\times\mathbf{r} + \operatorname{grad}\alpha_1, \quad \delta^2\mathbf{v} = \tfrac{1}{2}[\mathbf{f}\times\{\mathbf{fr}\}] + \operatorname{grad}\alpha_2$$

Correspondingly,

$$2\delta^2 E = \iiint \left[(\delta\mathbf{v})^2 + 2(\mathbf{v}\cdot\delta^2\mathbf{v})\right] dx = \iiint \left[(\delta\mathbf{v})^2 + \mathbf{v}\cdot\mathbf{f}\times\{\mathbf{fr}\} + 2\mathbf{v}\cdot\operatorname{grad}\alpha_2\right] dx$$

Integrating the last term by parts, we obtain the following form for the second variation of the energy E on the "sheet" F of the fields having equivorticity with $\mathbf{v}$, in terms of the variables $\mathbf{f}$ introduced in Section 4 :

$$2\delta^2 E = \iiint (\delta\mathbf{v})^2 + \mathbf{v}\times\mathbf{f}\{\mathbf{f}\cdot\mathbf{r}\}\,dx \tag{5.1}$$

This expression is quadratic with respect to $\mathbf{f}$, since $\delta\mathbf{v}$, linearly expressed in terms of $\mathbf{f}$, is

$$\delta\mathbf{v} = \mathbf{f}\times\mathbf{r} + \operatorname{grad}\alpha_1$$

where α_1 is determined from $\operatorname{div}\delta\mathbf{v} = 0$ and $(\delta\mathbf{v}\cdot\mathbf{n})|_\Gamma = 0$ and, therefore, is linearly dependent on $\mathbf{f}$. We also observe that according to Formula (2.3) $\{\mathbf{fr}\} = \delta\mathbf{r}$.

The following theorem is the analog of Theorem 1.2.

Theorem 5.1. If the quadratic form (5.1) is of fixed sign, then the flow $\mathbf{v}$ is stable with respect to small finite perturbations. By a small perturbation here is meant one of which $\delta\mathbf{v}$ and $\mathbf{f}$, i.e. $\delta\mathbf{v}$ and $\delta\mathbf{r}$ or form $|\delta^2 E|$, are small.

The form (5.1) represents the first integral to the linear problem of small oscillations close to a steady-state flow $\mathbf{v}$. In accordance with Theorem 1.3, the spectrum of this problem is symmetric with respect to both axes. Hence –

T h e o r e m 5.2. If some perturbation of the steady-state flow **v** is damped, then some other perturbation is amplified and the flow **v** is unstable.

The author was not able to find a flow **v** for which the quadratic form $\delta^2 E$ was of fixed sign for three-dimensional perturbations. However, in specifically symmetric cases Theorem 5.1 gives simple stability criteria.

6. Supplementary integrals. Generalizing Theorem 1.2, we shall assume that the Euler equation (1.2) has a first integral M such that for a steady-state flow **v**

$$\delta M = \iiint \mathbf{A}\cdot\delta\mathbf{v}\,dx \qquad (\mathbf{A}\times\operatorname{rot}\mathbf{v} = \operatorname{grad}\alpha) \tag{6.1}$$

The assumption (6.1) is satisfied in the following three examples.

E x a m p l e 6.1. For the energy integral $M_1 = E$ we have

$$\mathbf{A} = \mathbf{v} = [\mathbf{v}\times\operatorname{rot}\mathbf{v}] = \operatorname{grad}\lambda$$

according to (2.2).

E x a m p l e 6.2. If the domain D and the flow **v** are invariant with respect to displacements along the x-axis, the integral

$$M_2 = \iiint \mathbf{v}\cdot\mathbf{e}_x dx\,dy\,dz$$

is then preserved.

For it $A = \mathbf{e}_x$ and $\mathbf{A}\times\operatorname{rot}\mathbf{v} = \operatorname{grad}(\mathbf{v}\cdot\mathbf{e}_x)$.

E x a m p l e 6.3. If the domain D and the flow **v** are invariant with respect to rotations about the z-axis, then

$$M_3 = \iiint (\mathbf{v}\times\mathbf{R}\cdot\mathbf{e}_z)\,dx\,dy\,dz$$

is preserved, where **R** is the radius vector of the point x, y, z . In this case

$$\mathbf{A} = \mathbf{R}\times\mathbf{e}_z, \qquad \mathbf{A}\times\operatorname{rot}\mathbf{v} = \operatorname{grad}(\mathbf{v}\times\mathbf{R}\cdot\mathbf{e}_z)$$

T h e o r e m 6.1. The value of the integral M over the velocity field of a steady-state flow **v** is an extremal in comparison with the values over close equivorticity fields , provided that M satisfies condition (6.1).

The proof is identical to the proof of Theorem 1.2. The corresponding formulas of the second variation have the form

$$2\delta^2 M_2 = \iiint (\mathbf{e}_x\times\mathbf{f})\{\mathbf{f}\mathbf{r}\}\,dx\,dy\,dz \tag{6.2}$$

$$2\delta^2 M_3 = \iiint (R\times\mathbf{e}_z\times\mathbf{f})\{\mathbf{f}\mathbf{r}\}\,dx\,dy\,dz \tag{6.3}$$

Fixed sign behavior of some linear combination

$$\lambda_1\delta^2 M_1 + \lambda_2\delta^2 M_2 + \lambda_3\delta^2 M_3$$

is sufficient for the stability of **v** .

We shall illustrate the application of Theorem 5.1, Formulas (5.1),(6.2) and (6.3) in an example of plane flows.

7. Plane perturbations of plane flows. Let the flow $\mathbf{v}$ have a stream function $\psi(x,y)$ such that

$$\mathbf{v} = \psi_y, \quad -\psi_x, \quad 0; \qquad \mathbf{r} = 0, \quad 0, \quad -\Delta\psi \tag{7.1}$$

Substituting (7.1) into (5.1), we obtain after a brief calculation, taking into consideration that $\{\mathbf{fr}\} = \delta\mathbf{r}$, the formula given in [1]

$$2\delta^2 E = \iint \left[(\delta \mathbf{v})^2 + \frac{\nabla\psi}{\nabla\Delta\psi}(\delta\mathbf{r})^2\right] dx\, dy \tag{7.2}$$

From (7.2) and Theorem 5.1 there follows

C o r o l l a r y 7.1 (cf.[1]). In any domain plane flows with a concave velocity profile $(\nabla\psi / \nabla\Delta\psi > 0)$ are stable with respect to finite plane perturbations.

We shall refer to specifically symmetric flows. The case of plane-parallel flows (the Rayleigh theorem) is considered in detail in [1]. We shall consider the flow in the annulus between concentric circles. After a brief calculation, Formula (6.3) is reduced in the plane case to the form

$$2\delta^2 M_3 = \frac{1/2 \nabla R^2}{\nabla\Delta\psi}(\delta\mathbf{r})^2 \tag{7.3}$$

From (7.3) and Theorem 5.1 there follows

C o r o l l a r y 7.2. A plane circular flow in a circular annulus is stable with respect to small finite plane perturbations if the vorticity varies monotonously with the radius.

Actually, if the sign of $\nabla R^2 / \nabla\Delta\psi$ is preserved, then the form

$$\delta^2 H = \delta^2 E + \lambda\delta^2 M_3$$

is positive definite for suitable λ.

Finally, we note that the investigation of parallel flows with a single inflection point carried out in [1], owing to Formula (7.3), remains in effect for the case of circular flows.

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